The vibrating-membrane problem - based on basic principles and simulations

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Abstract

Rectangular and circular membranes have been modelled as discrete arrays of mass points connected by massless springs. Based on Newton's principles and Hooke's law, the movement of such membranes has been simulated. All vibrational modes, as known from closed form solutions of the corresponding wave equations, can be excited, with deviations from theoretical values of no more than a few percent. This approach can be used to develop an intuitive understanding of vibrating membranes. The phenomenon of regular vibrational modes provides a suitable starting point for a thorough mathematical treatment.

In a more general sense this topic demonstrates the possibility that elasticity is no longer a matter of high mathematical demand. The true nature of the "rigid body" as an unrealistic but perfect model can convincingly be demonstrated.

1. Introduction

As has been demonstrated recently, the vibrating-membrane problem can be used as a rather appropriate example to demonstrate the power of computer algebra systems (CAS) like Axiom Maple, Mathematica, Derive etc. [1].

This approach, however, depends on a well-developed mathematical ability on the part of the learner and on his or her willingness to accept such an abstract and demanding path of explanation, where the solution of differential equations serves as a description of real world phenomena, in this case the vibrating modes of an elastic membrane.

In the following we would like to show that the same results can be achieved with much less mathematical effort and in a more direct fashion, based only on Newton's principles and linear elastic forces.

2. Theoretical Background

Our system consists of a plane membrane, in principle of any shape, homogeneously stretched by a tension T, given as force per unit length. The membrane has a mass μ per unit area and the boundary is clamped.

For small vibrations and in the absence of external forces the wave equation, describing the motion of the different points (coordinates *x, y* in the plane of the membrane), is [2]:

$$\nabla^2 s = \frac{\mu}{T} \cdot \frac{\partial^2 s}{\partial t^2} = \frac{1}{v^2} \cdot \frac{\partial^2 s}{\partial t^2}$$

∇^2 is the Laplace operator, $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$ in rectangular co-ordinates *x, y*, and $v \equiv \sqrt{T/\mu}$ is the velocity of the waves in the elastic membrane. We have denoted by *s(x,y,t)* the transverse displacement of any point relative to the position when the membrane is at rest.

For membranes held along the edge (s=0, as boundary condition), we have to find standing-wave solutions of the wave equation which have nodes along the boundary of the membrane. For simple shapes (rectangular or circular membranes), the standing wave solutions or normal modes of vibration are usually worked out using a set of curvilinear coordinates in which the edge of the membrane forms one of the coordinate axes. In many cases we can use separation of variables which simplifies the problem.

In the following the main characteristics of the modes for the rectangular and circular membranes are described. With our simulation tool xyZET [3] we can in principle experiment with membranes of any shape. The results in this article, however, are restricted to rectangular and circular geometries which allows us to compare our simulated results with theoretical solutions of the related wave equation.

Rectangular Membrane (borders fixed: s=0 in x=0,a and y=0,b)

By separating the variables ($s=X(x)Y(y)\exp(i\omega t)$), the standing wave modes for this case can be expressed as follows:

$\sin(k_x x) \cdot \sin(k_y y)$, multiplied by a harmonic time dependence $\sin(\omega_0 t)$, where the resonance frequency, ω_0, will depend on the mode of vibration (values of k_x, k_y)

$$\left(\frac{\omega_0}{v}\right)^2 = k_x^{\ 2} + k_y^{\ 2}.$$

The boundary conditions require that k_x, k_y can have only the following values: $k_x = \frac{m\pi}{a}$, $k_y = \frac{n\pi}{b}$, where m,n (the mode indexes) can take only integer values.

The resonance frequency for this (m,n) mode will be

$$\left(\frac{\omega_{mn}}{v}\right)^2 = k_{xm}^{\ 2} + k_{yn}^{\ 2} = \left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2$$

with $v \equiv \sqrt{T/\mu}$.

Circular Membrane (border fixed: s=0, for r=a)

In this case we can use polar co-ordinates (r,θ). The spatial part of the wave function will be of the form $R(r)\Theta(\theta)$. The boundary conditions will act specifically on $R(r)$ which will be a Bessel function $J_m(kr)$ with zeros at well known (tabulated) values x_{mn} (m for the function and n for the nth. zero).

This leads to the relation $k_{mn}a=x_{mn}$ to force a zero at r=a, the radius of the membrane. This results in the following relation for computing the angular frequency associated with the different modes:

$\omega_{mn} = \frac{x_{mn}v}{a}$, where *v* is the velocity of the wave in the membrane.

The solution of our problem for the mode (m,n) is, basically, of the form:

$$s(r, \theta, t) = J_m(k_{mn}r)\cos(m\theta)\cos(\omega_{mn}t).$$

A dependence with $\sin(m\theta)$ is also possible, giving rise to the existence of 2 degenerate modes for each m (except for m=0). In general, a linear combination of both modes will be excited.

3. The simulation program xyZET

At IPN, a simulation program, named xyZET, has been developed whose key feature is the visualization of interacting objects in 3d [1] [3] [4]. The effects of all classical forces can be simulated.

The implemented algorithm is force based. For each single particles of all those placed within the cube, the sum of all applied forces is determined. By integrating Newtons second law stepwise, the acceleration, the change in velocity and the resulting displacement is calculated and displayed by deleting and redrawing the particles at its new positions.

Figure 1 shows circular and rectangular membranes as modelled in xyZET, where the particles at the border are fixed and all particles are connected by springs with their nearest neighbours.

Placing particles and connecting particles by springs is done by repetitive mouse clicks. The cube which surrounds the objects can be rotated to show the system from different perspectives.

All relevant parameters such as charge, mass, spring constant and spring length can be set and an external electric field can be simulated, changing in time and with variable intensity, period and direction.

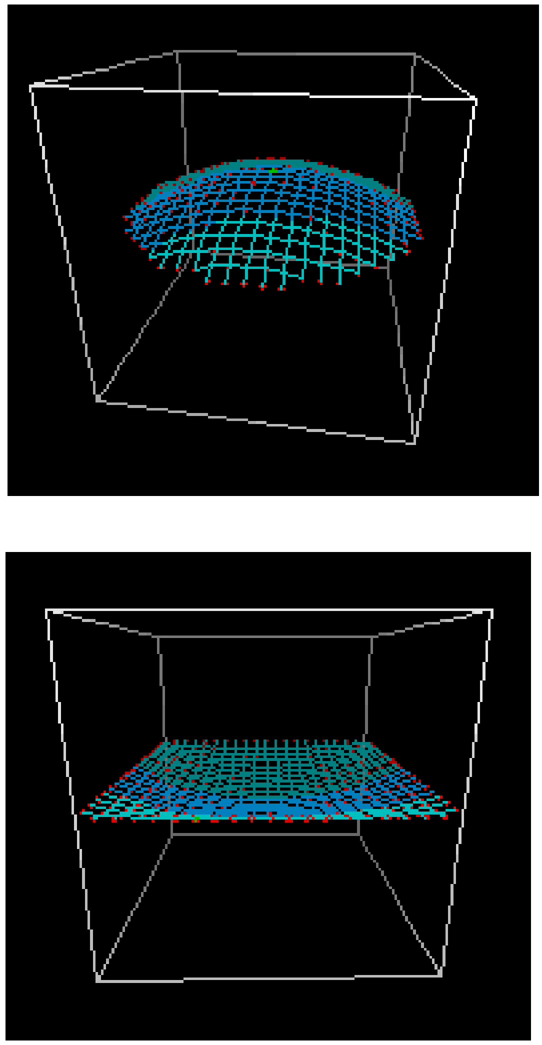

Figure 1 Membranes as modelled within xyZET

4. Experiments

To compare the simulation results from xyZET with those predicted theoretically, we have experimented with rectangular and circular membranes.

To do this, the mechanical characteristics of the membrane like tension T, and density μ, have to be determined. This information can be obtained from data available within xyZET.

1. Demo version download: http://www.ipn.uni-kiel.de/english/projekte/a7/a7.1/xyzet/mainpage_e.html

Once these values have been measured, the wave velocity, v, in the membrane, the eigenvalue and eigenfunction (resonant frequency and spatial distribution) for every mode can be computed as shown in the previous paragraph.

Results for a rectangular membrane

The membrane we used was made up of 21x21 particles, homogeneously distributed in a rectangular grid. From the measured values for tension T and density μ the wave velocity for mode 11, 21 and 33 was computed as well as the resonant frequencies ω_0 of the different modes.

By charging a few single particles, positioned at symmetry points of the expected mode and applying an external electric alternating field with ω_0 as frequency, the corresponding mode can be excited.

The agreement between the calculated resonant frequencies and the one measured with xyZET is between 3 and 5% for the lowest order modes. The spatial distribution of some of these modes are shown in Figure 2.

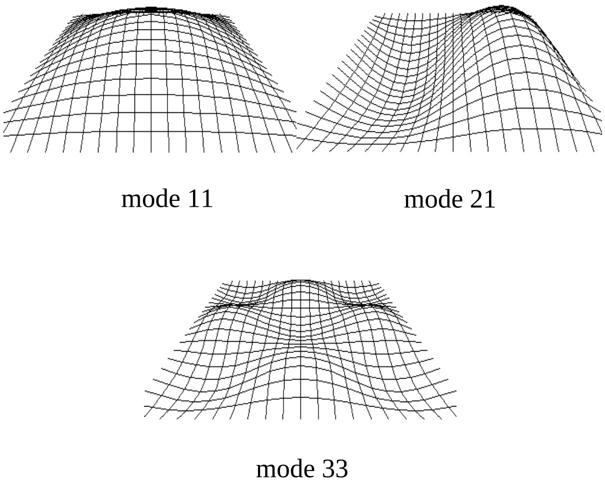

Figure 2 Display of the vibrating membrane for different modes

Results for a circular membrane

The membrane we used was again made up of 21x21 particles, homogeneously distributed over a circular area. The modes displayed in Figure 3 were excited and its velocity compared with the theoretical values.

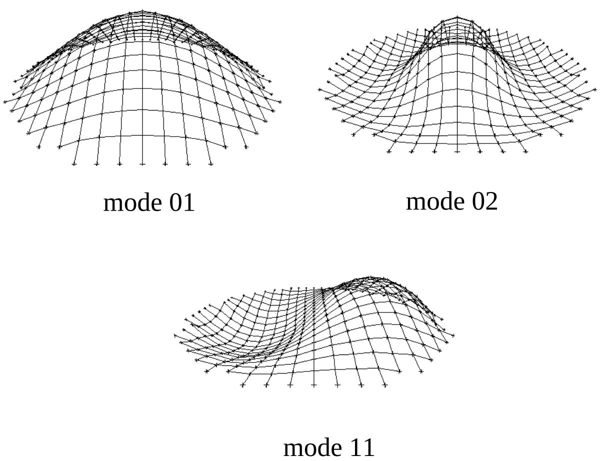

Figure 3 Display of a circular membrane vibrating in different modes

The differences between the simulated and theoretical results was always less than 6%.

5. Discussion

Didactical aspects

The topic “vibrating membranes” is a specific one and primarily only of interest for a specialised branch of engineering. For lectures in physics this topic is usually left aside due to the high demands of mathematics needed and the experimental difficulties to demonstrate the regularities of different vibrating modes.

Both these limiting factors have vanished. The power of modern computers allows to demonstrate all kinds of regularly or irregularly shaped membranes in their different vibrating modes in an effortless way. The question therefore has to be posed if this topic has some general didactical value and relevance.

We see two aspects: 1. With the support of modern computers the behaviour of extended elastic objects can easily be integrated in the physics curriculum. Nowadays extended objects are most of the time treated as rigid, which implies certain problems [5]. The model of the rigid body is an artificial one, neglecting internal processes and relying on non-causal distributions of forces. The treatment of extended bodies in physics could therefore be enriched if such objects would not only be presented as rigid but also as elastic - their real and only nature.

2. The whole is more than the sum of its parts. This basic statement can be visualised in a rather convincing and surprising way by applying our method, described above. When exciting for instance a single point in the centre of the rectangular membrane (fig 1 below) with an arbitrary frequency, some irregular vibrations of the complete membrane are displayed and regular pattern, if at all visible, show up only for short moments in time. But if the frequency is one of the eigenvalues of the membrane, the tiny vibrations around the excited particle at the centre slowly but irresistibly transform to a vibrating mode which controls every single particle of the membrane in a coordinated way. Such a mode is a property of the complete system. It cannot be derived from properties of its parts and it is more than the sum of all these individual properties.

We dare to mention that our swinging membranes are not only correctly represented but are nice and attractive to look at. This fact cannot be a substitute for learning physics, but it will never harm and may be more important for motivation than often acknowledged.

Besides these general aspects a more specific point deserves to be mentioned: the good agreement between a simulated membrane, modelled as a system of discrete parts and the theory, based on a continuous distribution of matter. This aspect is discussed in the following paragraph.

Explanation of the difference between theory and simulation

Results from theory and our simulation differ due to a number of factors.

First, the measurement of the resonance frequency carries an experimental error due to the method used. Resonance is detected by observing the shape of the space distribution of the vibrating elastic plane, and although this shape and the associated amplitude are very sensitive to frequency variations, we estimate an error in this measurement of the order of 1%.

Second, our model is a discrete one while the theory is based on a continuous mass distribution. Since at vibrating modes of higher order the spatial distribution of mass varies more strongly, the difference between the “continuous” theory and the discrete model should increase with modes of higher order. For a linear string, modelled by elastically connected mass points with the same spatial distribution as our plane, we computed this expected tendency with a maximum deviation of 1% for the 3rd. order mode (for a string made up of 21 particles).

A third reason for the difference between theory and experiment can be found in the fact that the theory does not take into account the variation of tension in time within any swinging plane. Such an idealization is valid only for rather small amplitudes. The theory also does not take into account the fact that for larger amplitudes the displacements do not only occur perpendicularly but also to a small degree in parallel to the plane. Since we need larger amplitudes to measure the resonance frequency, it cannot be expected that our simulated model results in the same values as those derived from the idealized theory.

Finally, the membranes we used in our simulation with circular planes did not have a precisely circular shape. This also may explain some of the difference between theory and experiment.

Basic laws and numerical solutions as added value

Traditional methods for teaching topics like "oscillations and waves" are characterized by doing experiments and then solving the corresponding differential equations. Experiments should be carried out whenever possible, and a thorough theoretical treatment is necessary. Depth and direction of this theoretical treatment, however, are open to discussion.

Besides looking only for closed form solutions of wave equations in 2 dimensions (Bessel functions in cylindrical coordinates), a more direct and much simpler path is now opened by starting from Newton’s basic principles and Hooke’s law and by looking for the corresponding numerical solutions, visualized on a computer screen.

Furthermore, this approach allows for a broad spectrum of exploratory actions. Direct feedback is received when changing the shape of the plane or internal parameters such as mass distribution and tension. This offers the possibility of building up an intuitive knowledge base about the behaviour of membranes as a starting point for the mathematical treatment.

We therefore see this approach not as an alternative but as an enrichment to the traditional method. The relation between cause, condition and effect is shown in a more direct manner and is offered for experimental exploration. Furthermore the comparison between our numerical solutions and the closed form solutions offers the opportunity to develop methodological knowledge of higher order.

6. References

[1] R. Portugal, L. Golebiowski, D. Frenkel, *Oscillation of membranes using computer algebra*, Am. J. Phys. 67 (6), 534-537 (June 1999).

[2] J. C. Slater and N. H. Frank, *Mechanics*, McGraw-Hill, NY, 1947.

[3] H. Härtel, *xyZET - A Simulation Program for Physics Teaching.* Journal for Science and Education Technology, 9, 3, 275-286, 2000

[4] H. Härtel, M. Lüdke, *3D-Simulations of Interacting Particles*, Computing in Science & Engineering, 4, 87-90, 2000

[5] H. Härtel, From simple to complex. But what is simple and for whom? This edition.